ELSEVIER
Journal of Geometry and Physics 41 (2002) 312–329
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A Fourier transform for sheaves on real tori Part II. Relative theory

U. Bruzzo [a,b,*], G. Marelli [b], F. Pioli [b]

[a] *Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Beirut 4, 34013 Trieste, Italy*
[b] *Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy*

Received 1 June 2001; received in revised form 24 July 2001

Abstract

If X is a symplectic family of Lagrangian tori, the dual family $\hat{X}$ has a natural complex structure. We define, for any dimension of X, a Fourier transform which yields a bijective correspondence between local systems supported on Lagrangian submanifolds of X and holomorphic vector bundles supported on complex subvarieties of $\hat{X}$ (suitable conditions being verified on both sides).

Subj. Class.: Differential geometry

MSC: 58J22; 53D12

Keywords: Fourier–Mukai transforms; Lagrangian submanifolds; Local systems; Holomorphic bundles

1. Introduction

The idea that, in accordance with the Strominger–Yau–Zaslow conjecture [21], a kind of Fourier–Mukai transform should describe transformation properties of D-branes under string-theoretic mirror symmetry dates back to 1996 [8]. The original Fourier–Mukai transform, mapping coherent sheaves on an abelian variety X to coherent sheaves on the dual variety $\hat{X}$, was introduced in [18]. A relative Fourier–Mukai transform for elliptic varieties was developed in [4–6,14] and was shown to describe a correct D-brane transformation pattern in the case of K3 surfaces [4,3]. An analogous result was shown to hold for elliptic Calabi–Yau three-folds in [1].

In the case of Calabi–Yau three-folds which are fibred in (special Lagrangian) real 3-tori, a similar description should be provided by a “real” relative Fourier transform. The presence

* Corresponding author.
E-mail addresses: bruzzo@sissa.it (U. Bruzzo), marelli@sissa.it (G. Marelli), pioli@sissa.it (F. Pioli).

PII: S0393-0440(01)00065-1

of singular fibers raises here a big problem because it is not clear how to handle them. As a first step, one may consider the simplified case when there are no singular fibers. In [7] an "absolute" version of such a transform was introduced (see also [2]). Here we begin the study of a relative version of that theory.

If X is a symplectic family of smooth Lagrangian tori, the dual family $\hat{X}$ has a natural complex structure. Then the relative Fourier transform yields a correspondence between local systems supported on Lagrangian submanifolds of X and holomorphic vector bundles supported on complex subvarieties of $\hat{X}$, where both sets of data satisfy suitable conditions (cf. later in this section). Some results along these lines were already contained in [2] but we strengthen and extend them considerably (cf. also [16]). We also carefully spell out the conditions on the submanifold S of X which ensure that the support of the transformed sheaf is a complex submanifold of $\hat{X}$.

The correspondence we get closely resembles Fukaya's homological mirror symmetry [11]. Comparison with that approach suggests that in order to extend the results presented in this paper to more general Lagrangian submanifolds (e.g. when the Lagrangian submanifold S is ramified over the base of the fibration X), or to the situation when X has singular fibers, it is necessary to allow for some kind of quantum corrections. Future extensions of the theory should also allow for the inclusion of a B-field and should investigate the possibility of describing the correspondence between the Floer homology of X and the algebraic cohomology of $\hat{X}$ in terms of the Fourier–Mukai transform studied in this paper. One could also study the relation between the transform presented in this paper and the constructions of Laumon [15] and Rothstein [20]; the local systems we transform are $\mathcal{D}$-modules, the same objects considered by these authors. This might also relate to possible applications to generalizations of the Krichever correspondence along the lines of [19].

We describe now the contents of this paper. In Section 2, we consider again the absolute case and complete the paper [7] by studying the transformation of a local system supported on an affine subtorus of a given torus. Section 3 starts our investigation of the relative case by setting up the general framework. In considering local systems supported on a Lagrangian submanifold of a symplectic torus fibration $X \to B$, we first analyze the two extreme cases, i.e. when the submanifold is a fiber of X and when it is a Lagrangian section of X (Section 3.1). In the second case, one finds that the transform is a holomorphic hermitian vector bundle on $\hat{X}$ which is flat along any fiber of $\hat{X}$, i.e. we get what may be called a holomorphic family of flat vector bundles. This sets up a bijective correspondence between local systems supported on Lagrangian sections of X and holomorphic bundles with unitary compatible, flat along the fiber directions of $\hat{X}$ (and satisfying some further conditions).

The intermediate, non-transversal cases (i.e. when one considers a Lagrangian submanifold $S \subset X$ whose projection onto B has a dimension strictly between 0 and $\dim B$) are more involved, and are analyzed in Section 4. To get a well-behaved transform one needs to assume, loosely speaking, that S intersects the fibers X_b of X (here $b \in B$) along subtori S_b of X_b, and that the vertical tangent spaces to S undergo parallel displacement under the natural Gauss–Manin connection defined in TX. Under this condition the transform of a local system on S is a holomorphic vector bundle supported on a complex submanifold of $\hat{X}$.

These result hold true whatever is the dimension of X, and do not require X to be Calabi–Yau (and not even complex). One should note that, when X is a Calabi–Yau manifold, the additional condition on the support S we have previously described is in general quite unrelated to the condition of S being special (in addition to being Lagrangian), and coincides with the latter only when X is complex three-dimensional, and the projection of the Lagrangian submanifold onto the base is (real) one-dimensional (this corresponds to a transformed sheaf which is a line bundle supported on a curve in $\hat{X}$). It is not clear to the authors whether this situation has any implication or motivation in string theory.

In Section 6, we draw some conclusions, in particular we comment upon the relation of this construction to Fukaya's homological mirror symmetry.

Two Appendices contain two proofs that, sketchy as they are, are too lengthy to be included in the main text.

We have tried to keep this paper independent of [7] as much as possible but some knowledge of the results and notation of that paper will help the reader.

2. The absolute case again: subtori

In this section, we complete the paper [7] by describing the transformation of $U(1)$ local systems supported on affine subtori of a given g-dimensional torus $T = V/\Lambda$. We will denote by $\hat{T}$ the dual torus and by $\mathcal{P}$ the normalized Poincaré sheaf on $T \times \hat{T}$. By $U(1)$ local system on a differentiable manifold X we mean a (smooth) complex line bundle $\mathcal{L}$ on X with a flat $U(1)$ connection ∇. The sheaf $\mathfrak{L} = \ker\nabla$ is a locally free $\mathbb{C}$-module on X, and one has $\mathcal{L} = \mathfrak{L} \otimes_{\mathbb{C}} \mathcal{C}^{\infty}_{X}$. We shall use quite interchangeably the notations $\mathfrak{L}$ and $(\mathcal{L}, \nabla)$.

Definition 1. A subtorus of T is a subset $S \subset T$ of the form $S = W/W \cap \Lambda$, where W is k-dimensional linear subspace of V such that the lattice $W \cap \Lambda$ has rank k. An affine subtorus is a subset of the form $S + x$ for an element $x \in T$.

Let $\mathfrak{L}$ be a $U(1)$ local system supported on a k-dimensional affine subtorus S of T. Let p_S and $\hat{p}_S$ the natural projections of $S \times \hat{T}$ onto the factors. By restricting the sheaves $\mathcal{P} \otimes \Omega^{m,0}$ to the closed submanifold $S \times \hat{T} \subset T \times \hat{T}$ one obtains a complex [1]

$$\begin{aligned} 0 &\to \ker\nabla_1^{\mathcal{L}} \to p_S^*\mathcal{L} \otimes \mathcal{P}_{|S\times\hat{T}} \xrightarrow{\nabla_1^{\mathcal{L}}} p_S^*\mathcal{L} \otimes (\mathcal{P} \otimes \Omega^{1,0})_{|S\times\hat{T}} \\ &\xrightarrow{\nabla_1^{\mathcal{L}}} p_S^*\mathcal{L} \otimes (\mathcal{P} \otimes \Omega^{2,0})_{|S\times\hat{T}} \to \cdots . \end{aligned}$$

Proposition 1.

1. $R^j \hat{p}_{S,*} \ker\nabla_1^{\mathcal{L}} = 0$ *for all* $j \neq k$.
2. $R^k \hat{p}_{S,*} \ker\nabla_1^{\mathcal{L}}$ *is supported on a* $(g-k)$*-dimensional affine subtorus* $\hat{S}$ of $\hat{T}$, *which is normal to* S.
3. *if* $\mathcal{L}$ *is trivial, then* $\hat{S}$ *goes through the origin of* $\hat{T}$, *otherwise it is an affine subtorus translated by the element of* $\hat{T}$ *corresponding to* $\mathcal{L}^*$.

[1] Let us recall that p, $\hat{p}$ denote the projections of $T \times \hat{T}$ onto the factors; the sheaves $\Omega^{m,n}$ are defined as $p^*\Omega^m_T \otimes \hat{p}^*\Omega^n_{\hat{T}}$; and $\nabla_1^{\mathcal{L}}$ is the $\Omega^{1,0}$ component of the coupled connection $p^*\nabla \otimes 1 + 1 \otimes \nabla_{\mathcal{P}}$.

4. *The sheaf* $R^k\hat{p}_{S,*}\ker\nabla_1^{\mathcal{L}}$ *on* $\hat{S}$ *is a* $U(1)$ *bundle, and has a compatible flat connection which makes it into a* $U(1)$ local system $\hat{\mathfrak{L}}$.

Proof. The proof of this proposition in given in Appendix A. □

Let us describe the content of this proposition in local coordinates; while this is just simple linear algebra, the explicit equations we are going to write will help to understand the more complicated relative situation. Let $(y^1, \dots, y^g)$ be flat coordinates in T, $(w_1, \dots, w_g)$ the corresponding dual flat coordinates in the dual torus $\hat{T}$, and write the equation for the affine subtorus S in the form

$$\sum_{j=1}^{g} a^i_j\, y^j + \chi^i = 0, \qquad i = 1, \dots, g-k.$$

The equations $\sum_{j=1}^{g} a^i_j\, y^j = 0$ describe a corresponding "linear subtorus" S_0; the equations of the dual torus S_0^* may be written implicitly as

$$\sum_{j,\ell=1}^{g} a^i_j\, g^{j\ell}\, w_\ell = 0, \qquad i = 1, \dots, g-k,$$

where the constant functions $g^{j\ell}$ are the components of the natural flat metric on $\hat{T}$, or explicitly as

$$w_\ell = \sum_{m=1}^{k} \tilde{a}^m_\ell\, \xi_m, \qquad \ell = 1, \dots, g \tag{1}$$

for a suitable $k \times (g-k)$ matrix $\tilde{a}$. The specification of the local system $\mathfrak{L}$ corresponds to a choice of the parameters $(\xi_1, \dots, \xi_k)$ in Eq. (1). The support $\hat{S}$ of the transformed local system is given by equations

$$\sum_{j=1}^{g} \gamma^j_m\, w_j + \xi_m = 0, \qquad m = 1, \dots, k,$$

where γ^j_ℓ is a matrix satisfying $\sum_{j=1}^{g} \gamma^j_\ell\, a^i_j = 0$. The local system $\hat{\mathfrak{L}}$ is given by the point in $\hat{S}_0^*$ whose coordinates are the numbers χ^i.

The pair $(\hat{S}, \hat{\mathfrak{L}})$ is the *Fourier–Mukai transform* of the pair $(S, \mathfrak{L})$. Of course, we may perform the same transformation from $\hat{T}$ to T (in addition to the obvious replacements, one twists by $\mathcal{P}^\vee$ instead of $\mathcal{P}$), and we have:

Proposition 2. *The Fourier–Mukai transform of* $(\hat{S}, \hat{\mathfrak{L}})$ *is naturally isomorphic to the pair* $(S, \mathfrak{L})$.

Let $\mathbf{Loc}_k(T)$ be the category of $U(1)$ local systems supported on affine subtori of T of dimension k. Objects of this category are triples $(S, \mathcal{L}, \nabla)$ (where S is an affine subtorus in

T, $\mathcal{L}$ a line bundle on S, and ∇ a flat unitary connection on $\mathcal{L}$) modulo isomorphisms, i.e. modulo vector bundle isomorphisms which commute with the actions of the connections (the two line bundles having the same support). The space of morphisms between two objects $(S_1, \mathcal{L}_1, \nabla_1)$ and $(S_2, \mathcal{L}_2, \nabla_2)$ of $\mathbf{Loc}_k(T)$ is defined by taking into account that the intersection $S = S_1 \cap S_2$ is a (possibly empty) finite collection of (possibly zero-dimensional) affine tori R_i, and one sets

$$\mathrm{Mor}((S_1, \mathcal{L}_1, \nabla_1), (S_2, \mathcal{L}_2, \nabla_2)) = \oplus_i \mathrm{Mor}_\nabla((R_i, \mathcal{L}_1, \nabla_1), (R_i, \mathcal{L}_2, \nabla_2)),$$

where $\mathrm{Mor}_\nabla((R_i, \mathcal{L}_1, \nabla_1), (R_i, \mathcal{L}_2, \nabla_2))$ is the set of morphisms between $\mathcal{L}_{1|R_i}$ and $\mathcal{L}_{2|R_i}$ compatible with the connections ∇_1 and ∇_2. It is easy to check that the Fourier–Mukai transform yields an equivalence of categories

$$\mathbf{Loc}_k(T) \simeq \mathbf{Loc}_{g-k}(\hat{T}).$$

3. Relative case: the geometric setting

Let (X, ω) be a connected symplectic manifold admitting a map $f : X \to B$ whose fibers are g-dimensional smooth Lagrangian tori. We assume that f admits a Lagrangian section $\sigma : B \to X$; according to [9], this makes X isomorphic, as a symplectic manifold fibred in Lagrangian submanifolds, to a quotient bundle T^*B/Λ, where Λ is a Lagrangian covering of B. The symplectic form ω provides an isomorphism Vert $TX \simeq f^*T^*B$. We also have an identification $TB \simeq R^1 f_*\mathbb{R} \otimes \mathbb{C}^\infty_B$, and this endows TB with a flat, torsion-free connection ∇_{GM}—the Gauss–Manin connection of the local system $R^1 f_*\mathbb{R}$. The holonomy of this connection coincides with the monodromy of the covering Λ (indeed, the horizontal tangent spaces may be identified with the first homology groups of the fibers with real coefficients).

Let $\hat{X} = R^1 f_*\mathbb{R}/R^1 f_*\mathbb{Z}$ be the dual family, with projection $\hat{f} : \hat{X} \to B$. Dualizing the isomorphism Vert $TX \simeq f^*T^*B$, we get a new isomorphism Vert $T\hat{X} \simeq \hat{f}^*TB$; combining this with the splitting of the Atiyah sequence

$$0 \to \mathrm{Vert}\, T\hat{X} \to T\hat{X} \to \hat{f}^*TB \to 0$$

provided by the Gauss–Manin connection (which can be regarded as a connection on $T\hat{X}$), one has a splitting

$$T\hat{X} \simeq \hat{f}^*TB \oplus \hat{f}^*TB.$$

By letting $J(\alpha, \beta) = (-\beta, \alpha)$ this induces a complex structure on $\hat{X}$, such that the holomorphic tangent bundle to $\hat{X}$ is isomorphic, as a smooth bundle, to $\hat{f}^*TB \otimes \mathbb{C}$.

On X we consider local coordinates symplectic coordinates $(x^1, \dots, x^g, w^1, \dots, w^g)$ such that the w are dual coordinates of B, and for fixed values of the x, the y are flat coordinates on the corresponding torus (local action–angle coordinates). Analogously, we may consider on $\hat{X}$ local coordinates $(x^1, \dots, x^g, \dots, w^1, \dots, w^g)$ such that the w are dual coordinates to the y. Local holomorphic coordinates on $\hat{X}$ are given by $z^j = x^j + iw^j$.

In this relative context, it is natural to consider the fiber product $Z = X \times_B \hat{X}$ of the fibrations X and $\hat{X}$. We shall denote by p, $\hat{p}$ the projections of Z onto its factors. On Z there is a Poincaré bundle $\mathcal{P}$ which may be described in an intrinsic way, however, it is enough to say that $\mathcal{P}$ is a line bundle on $Z = X \times_B \hat{X}$ equipped with a $U(1)$ connection $\nabla_{\mathcal{P}}$ whose connection form may be written in a suitable gauge as

$$\mathbb{A} = 2i\pi \sum_{j=1}^{g} w^j \, \mathrm{d}y_j .$$

Moreover, $\mathcal{P}$ has the property that for every $\xi \in \hat{X}$, $\mathcal{P}_{|\hat{p}^{-1}(\xi)}$ is isomorphic to $\mathcal{L}_\xi$ (the line bundle parametrized by ξ) as a $U(1)$ bundle.

If S is a closed submanifold of X, we define $Z_S = S \times_B \hat{X}$, with projections p_S, $\hat{p}_S$ onto the two factors, and denote $\mathcal{P}_S = \mathcal{P}_{|Z_S}$. We consider the exact sequence [2]

$$0 \to \hat{p}_S^* \Omega^1_{\hat{X}^S} \to \Omega^1_{Z_S} \xrightarrow{r} \Omega^1_{Z_S/\hat{X}^S} \to 0 \tag{2}$$

which defines the sheaf $\Omega^1_{Z_S/\hat{X}^S}$ of $\hat{p}_S$-relative differentials. Here $\hat{X}^S = \hat{p}_S(Z_S)$, and S is assumed to be chosen so that $\hat{p}_S : Z_S \to \hat{X}^S$ is a smooth submersive map. This assumption will be tacitly understood in the remainder of this section, while in the later sections it will be automatically satisfied. The Gauss–Manin connection ∇_{GM} provides a splitting of this exact sequence.

Analogously, if $\hat{S}$ is a closed submanifold of $\hat{X}$, we have a split exact sequence

$$0 \to p_{\hat{S}}^* \Omega^1_{X^{\hat{S}}} \to \Omega^1_{Z_{\hat{S}}} \xrightarrow{\hat{r}} \Omega^1_{Z_{\hat{S}}/X^{\hat{S}}} \to 0 \tag{3}$$

which defines the sheaf $\Omega^1_{Z_{\hat{S}}/X^{\hat{S}}}$ of $\hat{p}_{\hat{S}}$-relative differentials. For every sheaf $\mathcal{E}$ of $\mathcal{C}^\infty_S$-modules endowed with a flat connection ∇, one defines the following differential operators:

1. The operator
$$\nabla^{\mathcal{E}} : p_S^*\mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^\bullet_{Z_S} \to p_S^*\mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^{\bullet+1}_{Z_S},$$
obtained by coupling the pullback of the connection ∇ with the connection of the Poincaré sheaf $\nabla_{\mathcal{P}}$.
2. The operators $\nabla^{\mathcal{E}}_r$, $\nabla^{\mathcal{E}}_{\hat{r}}$ obtained by composing $\nabla^{\mathcal{E}}$ with the projections r, $\hat{r}$ onto the relative differentials. One has $(\nabla^{\mathcal{E}}_r)^2 = (\nabla^{\mathcal{E}}_{\hat{r}})^2 = 0$.

We shall consider the higher direct images $R^i \hat{p}_{S,*} \ker \nabla^{\mathcal{E}}_r$, which are the cohomology sheaves of the complex

$$\hat{p}_{S,*}(p_S^*\mathcal{E} \otimes \mathcal{P}_S) \xrightarrow{\nabla^{\mathcal{E}}_r} \hat{p}_{S,*}(p_S^*\mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^1_{Z_S/\hat{X}^S}) \xrightarrow{\nabla^{\mathcal{E}}_r} \hat{p}_{S,*}(p_S^*\mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^2_{Z_S/\hat{X}^S}) \to \cdots \tag{4}$$

[2] If $f : X \to Y$ is a differentiable map between two differentiable manifolds, and $\mathcal{F}$ any sheaf on Y, we shall denote by $f^{-1}\mathcal{F}$ the sheaf-theoretic inverse image of $\mathcal{F}$; if $\mathcal{F}$ is a sheaf of $\mathcal{C}^\infty_Y$-modules, we shall denote by $f^*\mathcal{F}$ its inverse image as a sheaf of modules, i.e.

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{C}^\infty_Y} \mathcal{C}^\infty_X .$$

As in the usual theory of the Fourier–Mukai transform, it is convenient to introduce a WIT notion. [3]

Definition 2. The pair $(\mathcal{E}, \nabla)$ is said to be WIT_k if $R^i \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}} = 0$ for $i \neq k$.

Now we want to state a condition for the sheaves $R^j \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}}$ to admit a connection induced, so to say, by the part of the operator $\nabla^{\mathcal{E}}$ complementary to $\nabla_r^{\mathcal{E}}$. The splitting of the exact sequence (2) provided by the Gauss–Manin connection ∇_{GM} allows one to make a splitting

$$\nabla^{\mathcal{E}} = \nabla_r^{\mathcal{E}} + \hat{\nabla}^{\mathcal{E}}.$$

The $\hat{\nabla}^{\mathcal{E}}$ operator induces connections on the higher direct images $R^j \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}}$ provided it anticommutes with the operator $\nabla_r^{\mathcal{E}}$. The anticommutator $\nabla_r^{\mathcal{E}} \circ \hat{\nabla}^{\mathcal{E}} + \hat{\nabla}^{\mathcal{E}} \circ \nabla_r^{\mathcal{E}}$ may be regarded as an operator $p_S^*\mathcal{E} \otimes \mathcal{P}_S \to p_S^*\mathcal{E} \otimes \mathcal{P}_S \otimes \Omega^2_{Z_S}$ and as such it coincides with the restriction to Z_S of $1 \otimes \mathbb{F}$, where $\mathbb{F}$ is the curvature of the connection $\nabla_{\mathcal{P}}$ of the Poincaré bundle. As a consequence, we have:

Proposition 3. *Assume that the sheaf $\mathcal{E}$ is supported on a closed submanifold $S \subset X$, the sheaf $R^j \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}}$ is supported on a closed submanifold $\hat{S} \subset \hat{X}$, and the curvature operator $\mathbb{F}$ vanishes on $S \times_B \hat{S} \subset Z$. Then the operator $\hat{\nabla}^{\mathcal{E}}$ induces a connection on the sheaf $R^j \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}}$.*

Eventually, we may introduce the Fourier–Mukai transform we shall study in the remainder of this paper.

Definition 3. If the pair $(\mathcal{E}, \nabla)$ is WIT_k and satisfies the condition in Proposition 3, the pair $(\hat{\mathcal{E}}, \hat{\nabla})$, where $\hat{\mathcal{E}} = R^k \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}}$ and $\hat{\nabla}$ is the connection induced as in Proposition 3, is called the Fourier–Mukai transform of $(\mathcal{E}, \nabla)$.

We end this section with an easy lemma which is useful when checking if the WIT property holds for some sheaf and connection.

Lemma 1. *Let $(\mathcal{E}, \nabla)$ be a local system supported on a closed submanifold S of X which intersects every fiberX_b along a closed submanifold S_b. For every $j = 1, \dots, g$ there is a canonical isomorphism*

$$(R^j \hat{p}_{S,*} \ker\nabla_r^{\mathcal{E}})_{|\hat{X}_b} \xrightarrow{\sim} R^j \hat{p}_{S_b,*} \ker\nabla_1^{\mathcal{E}_b}, \tag{5}$$

where $b = \hat{p}(\xi)$, $\hat{p}_b : X_b \times \hat{X}_b \to \hat{X}_b$ is the projection onto $\hat{X}_b$ and $\mathcal{E}_b$ is the restriction of $\mathcal{E}$ to S_b.

[3] Let us recall that “WIT” stands for “weak index theorem”.

Proof. The restriction $(R^j\hat{p}_{S,*}\ker\nabla^{\mathcal{E}}_r)_{|\hat{X}_b}$ is defined as $\hat{j}_b^{-1}(R^j\hat{p}_{S,*}\ker\nabla^{\mathcal{E}}_r)\otimes_{\hat{j}_b^{-1}\mathcal{C}^\infty_{\hat{X}}}\mathcal{C}^\infty_{\hat{X}_b}$ (here $j_b : S_b \to X$ and $\hat{j}_b : \hat{X}_b \to \hat{X}$ are the natural inclusions). The result is proved by applying the topological base change [12] to the diagram. □

$$\begin{tikzcd} S_b \times \hat{X}_b \arrow[r, "j_b\times\hat{j}_b"] \arrow[d, "\hat{p}_b"'] & S \times_B \hat{X} \arrow[d, "\hat{p}"] \\ \hat{X}_b \arrow[r, "\hat{j}_b"] & \hat{X} \end{tikzcd}$$

3.1. Fibers and Lagrangian sections

In studying the transformation of local systems supported by Lagrangian submanifolds we start by considering the case where the submanifold is either a fiber or a Lagrangian section. The first case is the simplest to deal with. It is enough to consider the case rank $\mathfrak{L} = 1$, since the higher rank case reduces immediately to this. Let us notice that the isomorphism class of the local system $\mathfrak{L}^*$ singles out a point in $\hat{X}$, which we denote by $[\mathfrak{L}^*]$. Since $X_b \times_B \hat{X} \cong X_b \times \hat{X}_b$, we obtain the usual "tautological" property of the Fourier–Mukai transform.

Proposition 4. *The pair* $(\mathcal{L}, \nabla) \equiv \mathfrak{L}$ *is WIT$_g$, and the sheaf* $\hat{\mathcal{L}} = R^g\hat{p}_*\ker\nabla^{\mathcal{L}}_r$ *is isomorphic to the skyscraper* $\mathbb{C}([\mathfrak{L}^*])$.

Now we construct a transform for local systems supported on sections of $X \to B$. This will generalize the tautological correspondence that in the absolute case holds between skycrapers of length one on a torus and $U(1)$ local systems on the dual torus. The transform will produce holomorphic line bundles on $\hat{X}$ with compatible $U(1)$ connections which satisfy some further conditions.

Let $S \subset X$ be the image of a Lagrangian section of $X \to B$, and $\mathfrak{L} \equiv (\mathcal{L}, \nabla)$ a unitary local system on S.

Proposition 5.

1. *The pair* $(\mathcal{L}, \nabla)$ *is WIT$_0$.*
2. $\hat{\mathcal{L}} = \hat{p}_{S,*}\ker\nabla^{\mathcal{L}}_r$ *is a rank-one locally free* $\mathcal{C}^\infty_{\hat{X}}$ *-module.*

Proof. Both claims follows from Lemma 1 and the absolute case. □

Since $\mathbb{F}_{|S\times_B\hat{X}} = 0$ the conditions of Proposition 3 are met, so that $\hat{\mathcal{L}}$ carries a $U(1)$ connection $\hat{\nabla}$. Let us express this connection in (action–angle) coordinates. We write the local equations of S as $y_j = \epsilon_j(x)$; as S is Lagrangian, one has $\partial\epsilon_j/\partial x^m = \partial\epsilon_m/\partial x^j$. Moreover, the x can be thought of as local coordinates on S. If the connection form associated with the local system $\mathfrak{L}$ is $A = i\sum_{j=1}^g A_j(x)\,\mathrm{d}x^j$, with $\partial A_j/\partial x^\ell = \partial A_\ell/\partial x^j$, then $\hat{\nabla}$ may be represented by the connection form

$$\hat{A} = i\sum_{j=1}^g A_j(x)\,\mathrm{d}x^j - 2i\pi\sum_{j=1}^g \epsilon_j(x)\,\mathrm{d}w^j.$$

In these coordinates the components of the connection form $\hat{A}$ do not depend on the w. Moreover, both the horizontal and vertical part (with respect to the splitting given by the Gauss–Manin connection) are flat, and in particular, the restriction of $\hat{\nabla}$ to any fiber $\hat{X}_b$ of $\hat{X} \to B$ is flat.

Remark 1. The independence of the components $\hat{A}$ on the w can be stated invariantly in a variety of ways. For instance, one can use the fact that the zero-section of $\hat{X}$ makes the latter into a (trivial) principal T^g-bundle over B; then, $\hat{\nabla}$ commutes with the action of T^g on $\hat{X}$.

The Hodge components of curvature form $\hat{F}$ of this connection may be written—recalling that in the complex structure, we have given to $\hat{X}$ the coordinates $z^j = x^j + iw^j$ are complex holomorphic—as

$$\hat{F}^{2,0} = \frac{\pi}{2} \sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k} \, \mathrm{d}z^k \wedge \mathrm{d}z^j$$

$$\hat{F}^{0,2} = -\frac{\pi}{2} \sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k} \, \mathrm{d}\bar{z}^k \wedge \mathrm{d}\bar{z}^j$$

$$\hat{F}^{1,1} = \frac{\pi}{2} \sum_{k,j} \left(\frac{\partial \epsilon_k}{\partial x^j} + \frac{\partial \epsilon_j}{\partial x^k} \right) \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^j .$$

Since S is Lagrangian, we have $\hat{F}^{0,2} = \hat{F}^{2,0} = 0$, so that $\hat{\mathcal{L}}$ may be given a holomorphic structure compatible with the connection $\hat{\nabla}$. Moreover, we have

$$\hat{F}^{1,1} = \pi \sum_{k,j} \frac{\partial \epsilon_j}{\partial x^k} \, \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^j .$$

Definition 4. The Fourier transform of $(S, \mathfrak{L})$ is the pair $(\hat{\mathcal{L}}, \hat{\nabla})$.

4. The non-transversal case

The results in Section 3.1 can be generalized to local systems supported on Lagrangian submanifolds of X other than sections. This allows us to enlarge the "dual" category on which the inverse Fourier–Mukai transform (defined in Section 5) acts, to a category of sheaves with connection (satisfying some suitable conditions) supported by complex subvarieties of $\hat{X}$. Such sheaves arise naturally in Fukaya's treatment of mirror symmetry (see footnote 1). We shall be interested in transforming local systems supported on a submanifold S of X such that:

(C1) S is a Lagrangian subvariety of X;
(C2) the intersection $S_b = S \cap X_b$ of S with a fiber of X, when non-empty, is a (possibly affine) subtorus S_b of X_b whose dimension does not depend on b.

Let $\mathfrak{L}$ be a $U(1)$ local system on S and ∇ the corresponding flat connection on $\mathcal{L} = \mathfrak{L}\otimes_{\mathbb{C}}\mathcal{C}^{\infty}_X$. We define as before a Fourier–Mukai transform of the local system $(S, \mathcal{L}, \nabla^{\mathcal{L}})$ at the sheaf level as

$$\hat{\mathcal{L}} = R^m\,\hat{p}_{S,*}\,\ker\nabla^{\mathcal{L}}_r$$

where m is the dimension of the tori S_b. This definition is motivated by the following result.

Proposition 6. *Let $(S, \mathcal{L}, \nabla^{\mathcal{L}})$ be a local system supported on a Lagrangian variety S which fulfills the conditions C1 and C2. Then the sheaf $\mathcal{L}$ is WIT_m.*

Proof. It follows from Lemma 1 and Proposition 1. □

Lemma 1 and Proposition 1 also imply that after restriction to its support, $\hat{\mathcal{L}}$ is a line bundle. We shall now show that, under some suitable conditions on the support S, the transform $\hat{\mathcal{L}}$ is supported on a complex submanifold $\hat{S}$ of the dual family $\hat{X}$. More precisely, we assume:

(C3) the vertical tangent spaces of the family of subtori $\{S_b\}_{b\in f(S)}$ are parallelly transported by the Gauss–Manin connection ∇_{GM} regarded as a connection in TX.

This requirement can be translated into a more explicit form in terms of the action–angle coordinates (x, y) we have previously introduced, in that it amounts to the condition that the family of subtori $\{S_b\}$ can be written as

$$\sum_{j=1}^{g} a_i^j\, y_j + \chi_i = 0, \qquad i = 1, \dots, g-m$$

with the matrix a_i^j constant and the χ_i are local functions on B.

Lemma 2. *Conditions C1, C2 and C3 imply that $f(S)$ is a submanifold of B of dimension $k = g - m$, and that it can be parametrized by the first k action coordinates x^j.*

Proof. The first claim follows from the fact that the horizontal part of the tangent space to S has constant dimension; the second from the Lagrangian condition which implies that the local equations of $f(S)$ in B are linear in the action coordinates. □

Proposition 7. *Let $(S, \mathcal{L}, \nabla)$ be a local system supported on a Lagrangian submanifold S fulfilling the conditions C1 and C2. The condition C3 is satisfied if and only if the support $\hat{S}$ of the transform $\hat{\mathcal{L}}$ is a complex submanifold of $\hat{X}$.*

Proof. A proof is given in Appendix B. □

Remark 2. In our setting there is no constraint on the dimension of X, the latter space is assumed to be just symplectic, and we consider local systems supported on Lagrangian submanifolds of X. On the other hand, string-theoretic mirror symmetry assumes, on physical grounds, that X is a (usually three-dimensional) Calabi–Yau manifold, and one considers special Lagrangian supports.[4] In this case, the condition that S is special Lagrangian implies, for $k = 1$, that the coefficients a_i^j are constant, so that this is a particular case within our treatment. On the contrary, for $k = 2$ the specialty property seems to be unrelated to the conditions that ensure the support $\hat{S}$ to be complex holomorphic.

Proposition 8. *Under the conditions of* Proposition 7, *the operator* $\hat{\nabla}^{\mathcal{L}}$ *(cf. Section 3) induces on* $\hat{\mathcal{L}}$ a $U(1)$ *connection.*

Proof. This will use the proof of Proposition 7 given in Appendix B. We know that $\hat{\nabla}^{\mathcal{L}}$ induces a connection on the Fourier–Mukai transform if the curvature $\mathbb{F}$ of the Poincaré bundle on $Z = X \times_B \hat{X}$ vanishes on $S \times_B \hat{S}$, where S and $\hat{S}$ are the supports of $\mathcal{L}$ and $\hat{\mathcal{L}}$, respectively. In view of the form of $\mathbb{F}$, this condition is met if for each $b \in B$ the intersections of S and $\hat{S}$ with the fibers X_b, $\hat{X}_b$ yield subtori of X_b, $\hat{X}_b$ that are normal to each other. But looking at the equations of the supports, Eqs. (B.1) and (B.4), and comparing with the absolute case (Proposition 2), we see that this condition is fulfilled. □

We shall now prove that $\hat{\mathcal{L}}$, as a line bundle on $\hat{S}$, has a holomorphic structure. Let $\hat{\nabla}$ be the connection induced on $\hat{\mathcal{L}}$.

Proposition 9. *If the support $\hat{S}$ of the transformed sheaf $\hat{\mathcal{L}}$ is a complex submanifold of $\hat{X}$, then $\hat{\nabla}$ induces a holomorphic structure on $\hat{\mathcal{L}}$.*

Proof. The connection 1-form of the connection ∇ can be written in an appropriate gauge as

$$A = i \sum_{j=1}^{k} \alpha_j(x^1, \dots, x^k)\, \mathrm{d}x^j + 2i\pi \sum_{\ell=1}^{g-k} \xi^\ell\, \mathrm{d}y_\ell,$$

with the quantities ξ^ℓ constant. From the proof of Proposition 1 given in the Appendix A, we know that the transformed connection $\hat{\nabla}$ is given in coordinates by the 1-form[5]

$$\hat{A} = -2i\pi \sum_{\ell=g-k+1}^{g} \chi_\ell(x^1, \dots, x^k)\, \mathrm{d}w^\ell + i \sum_{j=1}^{k} \alpha_j(x^1, \dots, x^k)\, \mathrm{d}x^j.$$

Rewriting this in terms of $w^1, \dots, w^k$ we obtain

$$\hat{A} = -2i\pi \sum_{\ell=g-k+1}^{g} \sum_{j=1}^{k} \chi_\ell(x^1, \dots, x^k)\, \tilde{\gamma}_j^\ell\, \mathrm{d}w^j + i \sum_{j=1}^{k} \alpha_j(x^1, \dots, x^k)\, \mathrm{d}x^j$$

[4] Let us recall that a special Lagrangian submanifold of a Calabi–Yau n-fold X is an oriented real n-dimensional submanifold Y which is Lagrangian with respect to the Kähler form of X, and such that the global trivialization Ω of the canonical bundle of X may be chosen so that its imaginary part vanishes on Y. For more details cf. [13].

[5] Also in this case, we find that the transformed connection $\hat{\nabla}$ satisfies the condition of Remark 1.

where the coefficients $\tilde{\gamma}_j^\ell$ are constant. Since $\mathrm{d}(\sum_j \alpha_j\,\mathrm{d}x^j) = 0$ because of the flatness of ∇, it follows that the curvature of $\hat{\nabla}$ is given by

$$\hat{F} = -2i\pi \sum_{\ell=g-k+1}^{g} \sum_{j,m=1}^{k} \frac{\partial \chi_\ell}{\partial x^j}\,\tilde{\gamma}_m^\ell\,\mathrm{d}x^j \wedge \mathrm{d}w^m.$$

Since $\tilde{\gamma}_m^{g-k+j} = \partial \zeta^{g-k+j}/\partial x^m$, where the functions ζ^{g+j} are those of the Eq. (B.1), the condition $\hat{F}^{0,2} = 0$ can be written as

$$\sum_{\ell=g-k+1}^{g} \left[\frac{\partial \zeta^\ell}{\partial x^j}\frac{\partial \chi_\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m}\frac{\partial \chi_\ell}{\partial x^j}\right] = 0, \qquad 1 \le j < m \le k.$$

But this is the system of Eq. (B.6), therefore, when S is Lagrangian, this condition is automatically satisfied. □

Remark 3. (The higher rank case): So far we have for simplicity considered only the transformation of local systems of rank one. However the higher rank case, under the same conditions, can be treated along the same lines, obtaining on the $\hat{X}$ side holomorphic vector bundles of the corresponding rank supported on complex submanifolds of $\hat{X}$.

5. Invertibility

In this section, we shall prove that the Fourier–Mukai transform we have defined inverts. However, we shall only discuss the inverse transform of rank 1 sheaves. The higher rank case requires to consider Lagrangian submanifolds of X which ramify over B, and this will be done in a further paper of this series.

We shall, therefore, consider a holomorphic line bundle $\hat{\mathcal{L}}$ supported on a k-dimensional complex submanifold $\hat{S}$ of $\hat{X}$, equipped with a compatible $U(1)$ connection $\hat{\nabla}$. Moreover, we shall assume that:

- (D1) $\hat{S}$ intersects the fibers of $\hat{X}$ along affine subtori of complex dimension k;
- (D2) the horizontal part of the connection $\hat{\nabla}$ is flat (horizontality is given by the Gauss–Manin connection);
- (D3) the connection $\hat{\nabla}$ is invariant under the action of T^g on $\hat{X}$ (cf. Remark 3).

These conditions allow us to write the local connection form of $\hat{\nabla}$ as

$$\hat{A} = i \sum_{j=1}^{k} \alpha_j(x^1,\ldots,x^k)\,\mathrm{d}x^j + 2i\pi \sum_{j=1}^{k} \beta_j(x^1,\ldots,x^k)\,\mathrm{d}w^j,$$

where the functions α_j satisfy (as a consequence of D2) the closure condition $\partial\alpha_j/\partial x^\ell = \partial\alpha_\ell/\partial x^j$. This shows that the restriction of $\hat{\nabla}$ to any fiber $\hat{X}_b$ of $\hat{X} \to B$ yields a flat connection on $\hat{\mathcal{L}}_{|\hat{X}_b}$.

Let $p_{\hat{S}}$, $\hat{p}_{\hat{S}}$ the canonical projections of $X \times_B \hat{S}$ onto its factors. We consider the operator

$$\nabla_{\hat{r}}^{\hat{\mathcal{L}}} = \hat{r} \circ (\hat{p}_{\hat{S}}^{*} \nabla^{\mathcal{L}} \otimes 1 + 1 \otimes \nabla_b \mathcal{P}_{\hat{S}}^{\vee})$$

and in terms of it we define a Fourier–Mukai transform from sheaves on $\hat{X}$ to sheaves on X (notice that we twist with the dual Poincaré bundle $\mathcal{P}^{\vee}$).

Proposition 10. *$\mathcal{L}$ is WIT$_k$, and $\mathcal{L} = R^k p_{\hat{S},*} \ker \nabla_{\hat{r}}^{\hat{\mathcal{L}}}$ is supported on a Lagrangian submanifold S of X such that every intersection $S_b = S \cap X_b$ is an affine subtorus of X_b of dimension $g-k$ (when non-empty). Moreover, the family of subtori S_b is parallelly transported by the Gauss–Manin connection ∇_{GM}. Finally, a flat connection ∇ is naturally induced on $\mathcal{L}$.*

Proof. The WIT condition follows immediately from Lemma 1. To show the remaining part of the claim we write local equations for $\hat{S}$ as

$$\begin{cases} x^{k+j} = \zeta^{k+j}(x^1, \ldots, x^k), & j = 1, \ldots, g-k \\ w^{k+j} = \sum_{i=1}^{k} P_i^{k+j}(x^1, \ldots, x^k)\, w^i + Q^{k+j}(x^1, \ldots, x^k), & j = 1, \ldots, g-k. \end{cases}$$

Performing a fiberwise transform we obtain the following equations for the support S of the transform $\mathcal{L}$:

$$y_l + \sum_{m=k+1}^{g} P_l^m(x^1, \ldots, x^k) y_m + \beta_l(x^1, \ldots, x^k) = 0,$$

where $l = 1, \ldots, k$. It remains to show that S is Lagrangian and that the family $\{S_b\}_{b \in \hat{f}(S)}$ is parallelly transported by the Gauss–Manin connection. The latter point follows from the complex structure of $\hat{S}$ (cf. Proposition 7): the Cauchy–Riemann equations for $\hat{S}$ imply that the coefficients P_l^{k+j} and Q^{k+j} are constant. As far as the Lagrangian property of S is concerned, the holomorphicity of $\hat{S}$ and $\hat{\mathcal{L}}$ imply the Eq. (B.2) in the proof of Proposition 7 (in Appendix B). Therefore, S is Lagrangian. Observe that the transformed connection ∇ has a 1-form given by

$$A = i \sum_{j=1}^{k} \alpha_j(x^1, \ldots, x^k)\, \mathrm{d}x^j - 2i\pi \sum_{m=k+1}^{g} Q^m\, \mathrm{d}y_m,$$

whence, we can immediately deduce its flatness. □

6. Conclusions

We summarize here the main results of this paper. We have shown that a suitably defined Fourier–Mukai transform $\mathcal{F}$ maps a $U(1)$ local system supported on a Lagrangian subvariety S of X satisfying the conditions C1–C3 (cf. Section 4) into a holomorphic line bundle $\hat{\mathcal{L}}$ supported a complex subvariety $\hat{S}$ of $\hat{X}$; moreover, $\hat{\mathcal{L}}$ is endowed with a $U(1)$ connection such that conditions D1–D3 (Section 5) are satisfied.

Conversely, if we start with a holomorphic line bundle supported on a complex subvariety $\hat{S}$ of $\hat{X}$ equipped with a $U(1)$ connection $\hat{\nabla}$ such that conditions D1–D3 are satisfied, we define a dual Fourier–Mukai transform $\hat{\mathcal{F}}$ that maps such objects into a $U(1)$ local system supported on a Lagrangian subvariety S such that conditions C1–C3 are fulfilled. The explicit forms of the two transforms we have written in Sections 3.1 and 4 show that the transforms are one the inverse of the other. This parallels the classical result in [18] and generalizes the one in [2], whose authors consider the case where X and $\hat{X}$ are S^1-fibrations over S^1 ($\hat{X}$ is actually an elliptic curve) and $\mathcal{L}$ is a local system on an affine line $S \subset X$. Observe that in this case the conditions C1–C3 and D1–D3 are trivially satisfied.

Finally, we would like to comment upon the relation of the construction we have described in this paper with Fukaya's homological mirror symmetry. First we notice that, in the absence of the B-field and with no singular fibers, our "mirror manifold" $\hat{X}$ coincides with Fukaya's, also taking into account its complex structure. Let S be a Lagrangian submanifold of X, and $\beta = (\mathcal{L}, \nabla)$ a local system on it. Fukaya proposes to construct on $\hat{X}$ a coherent sheaf whose fiber at a point $(b, \alpha) \in \hat{X}$ (where $\alpha = (L_\alpha, \nabla_\alpha)$ is a local system on the fiber X_b) is given by the Floer homology

$$HF^\bullet((X_b, \alpha), (S, \beta)).$$

This homology may be proved [10] to be isomorphic to

$$H^{\bullet-\eta(X_b,S)}(S \cap X_b, \mathcal{H}om_\nabla(\mathcal{L}_\alpha, \mathcal{L})),$$

where $\eta(X_b, S)$ is a Maslov index, and $\mathcal{H}om_\nabla(\mathcal{L}_\alpha, \mathcal{L})$ is the sheaf of ∇-compatible morphisms between $\mathcal{L}_\alpha$ and $\mathcal{L}$. It is not difficult to show that, up to a dual, this fiber is isomorphic to the fiber of our transform $\hat{\mathcal{L}}$. However, the concrete construction done in (see footnote 1) is not in terms of Floer homology, but it is an ad-hoc one, which may be compared with ours when $X = T^{2g}$, $B = T^g$ and S is a Lagrangian embedding of T^g. In this case, the vector bundle constructed on $\hat{X}$ coincides with ours.

It should be noted that our construction provides on the "mirror side" $\hat{X}$ more data, in that we obtain on $\hat{\mathcal{L}}$ a connection. It is interesting to note that this connection is not invariant under Hamiltonian diffeomorphisms of X, while the remaining geometric data on $\hat{X}$ are.

Acknowledgements

We are thankful to K. Fukaya and D. Hernández Ruipérez for useful discussions. This work has been partly supported by the Friuli-Venezia Giulia Region Research Project "Non-commutative geometry: algebraic, analytical and probabilistic aspects and applications to mathematical physics" and by a grant awarded to the third author by the University of Genova under the program "Finanziamenti a singoli e giovani ricercatori". The first and third authors are members of VBAC (Vector Bundles on Algebraic Curves), which is partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and EDGE (EC FP5 Contract no. HPRN-CT-2000-00101). The third author is a member of GNSAGA.

Appendix A

We provide here a sketch of the proof of Proposition 1. It involves a number of computations but it is conceptually very easy. For further details we refer to [17].

We first consider the case when S is a one-dimensional affine subtorus of T. The direct image $R^i\hat{p}_{S,*}(\ker\nabla_1^{\mathcal{L}})$ is by definition the sheaf associated to the presheaf $\hat{U} \quad H^i(S \times \hat{U}, \ker\nabla_1^{\mathcal{L}}) \simeq H^i\left(\Omega^{\bullet,0}(p_S^* L \otimes \mathcal{P}_S)(S\times\hat{U}), \nabla_1^{\mathcal{L}}\right)$. When $i=0$, take an element s in $H^0(S\times\hat{U}, \ker\nabla_1^{\mathcal{L}})$ and consider its restriction to $S\times\{y\}$, with $y\in\hat{U}$, which is a global section of $\mathcal{L}\otimes\mathcal{P}_{S\times\{y\}}$. This means that $\mathcal{P}_{S\times\{y\}}$ is non-trivial for every y in the complement of $\ker\psi$, which is a dense subset of $\hat{T}$ (here ψ is the natural map $\psi : \hat{T} \to \mathit{Hom}(\pi_1(S), U(1))$). The same holds for $\mathcal{L}\otimes\mathcal{P}_{S\times\{y\}}$. Since $\nabla_1^{\mathcal{L}} s_{|S\times\{y\}} = 0$ for every $y\in\hat{T}$, the restriction of s to $S\times\{y\}$ vanishes for y in a dense subset of $\hat{T}$, so that s vanishes everywhere.

When $i=1$ we need to write the equation of S explicitly. For simplicity we only give some details in the case $\dim T = 2$. Let (y^1, y^2) be flat coordinates on T and (w_1, w_2) flat dual coordinates on $\hat{T}$. We pick a gauge where the Poincaré bundle has an automorphy factor (cf. [7])

$$a_{\mathcal{P}}(y^1, y^2, w_1, w_2, \lambda^1, \lambda^2, \mu_1, \mu_2) = \mathrm{e}^{2\pi\mathrm{i}(\lambda^1 w_1 + \lambda^2 w_2)}.$$

The equation of S in the universal cover of T is given by an affine line $y^2 = ay^1 + b$. Let $A = \bar{y}_1\,\mathrm{d}y^1$ be the connection form of the local system $(\mathcal{L}, \nabla^{\mathcal{L}})$ on S. We need to compute $H^1(S\times\hat{U}, \ker\nabla_1^{\mathcal{L}})$. So take an element $\tau \in (\Omega^{1,0}(p_S^*\mathcal{L}\otimes\mathcal{P}_S)(S\times\hat{U}), \nabla_1^{\mathcal{L}})$. Observe that τ is closed with respect to $\nabla_1^{\mathcal{L}}$ because $\dim S = 1$. If we let

$$\tau = \phi(\xi, w_1, w_2)\,\mathrm{d}\xi$$

where ξ is the natural coordinate on S, the automorphy condition satisfied by τ can be expressed in the form

$$\phi(\xi + \sqrt{p^2+q^2}, w_1, w_2) = \mathrm{e}^{p(w_1+\bar{w}_1)+qw_2}\phi(\xi, w_1, w_2)$$

having set $a = q/p$ with q, p coprime.

Suppose that τ is exact so that we can write $\tau = \nabla_1^{\mathcal{L}} s$ where $s \in C^\infty(S\times\hat{U}, \ker\nabla_1^{\mathcal{L}})$. Then s can be written in the form

$$s(\xi, w_1, w_2) = \int_0^\xi \phi(u, w_1, w_2)\,\mathrm{d}u + c(w_1, w_2)$$

but this is well defined if and only if the automorphy conditions satisfied, and one can easily check that this amounts to

$$c(w_1, w_2)(1 - \mathrm{e}^{2\pi\mathrm{i}(p(w_1+\bar{w}_1)+qw_2)}) = -\int_0^{\sqrt{p^2+q^2}} \phi(u, w_1, w_2)\,\mathrm{d}u.$$

This equation may be solved for c in the complement of the set $\hat{S}$ defined by

$$w_2 = -\left(\frac{1}{a}w_1 - \frac{\bar{w}_1}{a}\right);$$

thus, arguing as in [7], we obtain that the support of $R^1\hat{p}_*\ker\nabla_1^{\mathcal{L}}$ is exactly $\hat{S}$.

To compute the sheaf $R^1\hat{p}_{S,*}\ker\nabla_1^{\mathcal{L}}$, we note that the map

$$\varpi : \Omega^{1,0}(p_S^*\mathcal{L}\otimes\mathcal{P})(S\times\hat{U}) \to \mathcal{C}^\infty(\hat{S}\cap\hat{U}), \quad \tau \mapsto -\int_0^{\sqrt{p^2+q^2}} \phi(u, w_1, w_2)\,\mathrm{d}u$$

is surjective: if $f \in \mathcal{C}^\infty(\hat{S}\cap\hat{U})$ and s is a section of the Poincaré bundle over $S\times\hat{S}$, then the 1-form $\tau = \phi\,\mathrm{d}\xi$ defined by

$$\phi(\xi, w_1, w_2) = \beta\, s(\xi, w_1)\, f(w_2),$$

with $1/\beta = -\int_0^{\sqrt{p^2+q^2}} s(u,0)\,\mathrm{d}u$, satisfies $\varpi(\tau) = f$ and the correct automorphy condition. Thus, $H^1(S\times\hat{U}, \ker\nabla_1^{\mathcal{L}}) = \mathcal{C}^\infty(\hat{S}\cap\hat{U})$.

The transformed sheaf is endowed with a flat connection induced by $\nabla_2^{\mathcal{L}}$, the (0, 1) part of the connection $p_S^*\nabla^{\mathcal{L}}\otimes 1 + 1\otimes\nabla_{\mathcal{P}_S}$, because the curvature of the Poincaré bundle restricts to zero on $S\times\hat{S}$.

Of course, $R^i\hat{p}_{S,*}(\ker\nabla_1^{\mathcal{L}}) = 0$ for $i > 1$ because S is one-dimensional. This proof is extended to the case $\dim S > 1$ by using a Künneth formula.

Appendix B

Here we prove Proposition 7. For notational convenience we suppose that $k \le g/2$; the complementary case $k > g/2$ can be treated similarly. In the action–angle coordinates x and y, we can write the local equations for S as

$$\begin{cases} y_{g-k+j} = \eta_{g-k+j}(x^1,\dots,x^k, y_1,\dots,y_{g-k}), & j = 1,\dots,k \\ x^{k+i} = \zeta^{k+i}(x^1,\dots,x^k), & i = 1,\dots,g-k. \end{cases} \tag{B.1}$$

Since S is Lagrangian one has

$$\begin{cases} \delta_j^m + \sum_{\ell=g-k+1}^{g} \dfrac{\partial\zeta^\ell}{\partial x^j}\dfrac{\partial\eta_\ell}{\partial y_m} = 0, & j, m = 1,\dots,k \\ \dfrac{\partial\zeta^{k+i}}{\partial x^m} + \displaystyle\sum_{\ell=g-k+1}^{g} \dfrac{\partial\zeta^\ell}{\partial x^m}\dfrac{\partial\eta_\ell}{\partial y_{k+i}} = 0, & i = 1,\dots,g-2k; m = 1,\dots,k \\ \sum_{\ell=g-k+1}^{g}\left[\dfrac{\partial\zeta^\ell}{\partial x^j}\dfrac{\partial\eta_\ell}{\partial x^m} - \dfrac{\partial\zeta^\ell}{\partial x^m}\dfrac{\partial\eta_\ell}{\partial x^j}\right] = 0, & 1\le j < m \le k. \end{cases} \tag{B.2}$$

The equations of the subtori S_b can be written in a linear form

$$y_{g-k+j} = \sum_{m=1}^{g-k} a_{g-k+j}^m(x^1,\dots,x^k)\, y_m + \chi_{g-k+j}(x^1,\dots,x^k),$$
$$j = 1,\dots,k. \tag{B.3}$$

To find the equations of $\hat{S}$, we shall perform a fiberwise transform and use the Künneth formula as in [7]. First we split every subtorus S_b as a product of one-dimensional tori $r_i(b)$ which have linear equations given by

$$\begin{cases} y_l = 0, & l = 1, \dots, g-k, \ell \neq i \\ y_{g-k+j} = a^i_{g-k+j}(x^1, \dots, x^k) y_i + \chi_{g-k+j}(x^1, \dots, x^k), & j = 1, \dots, k. \end{cases}$$

Observe that we can also split the local system $\mathcal{L}$ on S_b as a box product of local systems $\mathcal{L}_i(b)$ on $r_i(b)$ where $i = 1, \dots, g-k$. Transforming the local system $\mathcal{L}_i(b)$ on $r_i(b)$, we get the following equation for the support of $\mathcal{L}_i(b)$ (see Appendix A):

$$w^i + \sum_{\ell=g-k+1}^{g} \gamma^i_\ell(x^1, \dots, x^k) w^\ell + \xi^i$$

where the constant term ξ^i describes the automorphy of $\mathcal{L}_i$ (here i is fixed), and the matrix γ^i_ℓ satisfies the condition $\sum_{j=1}^{g} \gamma^j_\ell\, a^i_j = 0$. Then $\hat{S}$ is the intersection of the supports $\hat{r}_i$, so that its equations are of the form

$$w^{k+i} = \sum_{j=1}^{k} \tilde{\gamma}^{k+i}_j(x^1, \dots, x^k)\, w^j + \varsigma^{k+i}(x^1, \dots, x^k), \qquad i = 1, \dots, g-k \tag{B.4}$$

together with the second set of Eq. (B.1). Here we have solved with respect to $w^1, \dots, w^k$. These equations may be used to replace the functions η in Eq. (B.1), thus, getting

$$\begin{cases} \delta^m_j + \sum_{\ell=g-k+1}^{g} \dfrac{\partial \zeta^\ell}{\partial x^j} a^m_\ell = 0, & j, m = 1, \dots, k \\ \dfrac{\partial \zeta^{k+i}}{\partial x^m} + \displaystyle\sum_{\ell=g-k+1}^{g} \dfrac{\partial \zeta^\ell}{\partial x^m} a^{k+i}_\ell = 0, & i = 1, \dots, g-2k;\ m = 1, \dots, k \end{cases} \tag{B.5}$$

$$\sum_{\ell=g-k+1}^{g} \left[\frac{\partial \zeta^\ell}{\partial x^j} \frac{\partial \chi_\ell}{\partial x^m} - \frac{\partial \zeta^\ell}{\partial x^m} \frac{\partial \chi_\ell}{\partial x^i} \right] = 0, \quad 1 \leq j < m \leq k. \tag{B.6}$$

The solution of Eq. (B.5) is

$$\frac{\partial \zeta^{k+i}}{\partial x^j} = \tilde{\gamma}^{k+i}_j, \qquad j = 1, \dots, k, \quad i = 1, \dots, g-k. \tag{B.7}$$

If the submanifold S is Lagrangian, the conditions Eq. (B.7) admit solutions in ζ. We must check that the support $\hat{S}$ is holomorphic, i.e. the equations that define it fulfill the Cauchy–Riemann conditions. The latters are satisfied if and only if the coefficients $\tilde{\gamma}^{k+i}_j$ do not depend on the x, but this is true if and only if the coefficients γ^{k+i}_{g-k+1} are in turn independent of the x. As a result, we have proved that when S is Lagrangian, the tangent spaces to the S_b are parallelly transported by ∇_{GM} if and only if $\hat{S}$ is holomorphic.

One may note that the coefficients χ^j play no role in the specification of the complex structure of $\hat{S}$. Moreover, let us remark that Eq. (B.4) shows that the intersections of the support $\hat{S}$ with the fibers $\hat{X}_b$ are affine subtori.

References

[1] B. Andreas, G. Curio, D. Hernández Ruipérez, S.-T. Yau, Fourier–Mukai transform and mirror symmetry for D-branes on elliptic Calabi–Yau, Math. AG/0012196.
[2] D. Arinkin, A. Polishchuk, Fukaya category and Fourier transform, Math. AG/9811023.
[3] P. Aspinwall, R. Donagi, The heterotic string, the tangent bundle, and derived categories, Adv. Theor. Math. Phys. 2 (1998) 1041–1074.
[4] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, J.M. Muñoz Porras, Mirror symmetry on K3 surfaces via Fourier–Mukai transform, Commun. Math. Phys. 195 (1998) 79–93.
[5] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez, J.M. Muñoz Porras, Relatively stable bundles on elliptic surfaces, Preprint (1999).
[6] T. Bridgeland, Fourier–Mukai transforms for elliptic surfaces, J. Reine Angew. Math. 498 (1998) 115–133.
[7] U. Bruzzo, G. Marelli, F. Pioli, A Fourier transform for sheaves on real tori. Part I. The equivalence $\mathbf{Sky}(T) \simeq \mathbf{Loc}(\hat{T})$, J. Geom. Phys. 39 (2001) 174–182.
[8] M.R. Douglas, G. Moore, D-branes, quivers, and ALE instantons, Hep-th/9603167.
[9] J.J. Duistermaat, On global action–angle coordinates, Commun. Pure Appl. Math. 33 (1980) 687–706.
[10] K. Fukaya, Floer homology and mirror symmetry I, Preprint (1999), available from the web page http://www.kusm.kyoto-u.ac.jp/∼fukaya/mirror1.pdf.
[11] K. Fukaya, Mirror symmetry of Abelian varieties and multi-theta functions, Preprint (2000), available from the web page http://www.kusm.kyoto-u.ac.jp/∼fukaya/abelrev.pdf.
[12] S.I. Gelfand, Yu.I. Manin, Methods of Homological Algebra, Springer, Berlin, 1991.
[13] R. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982) 47–157.
[14] D. Hernández Ruipérez, J.M. Muñoz Porras, Structure of the moduli space of stable sheaves on elliptic surfaces, Math. AG/9809019.
[15] G. Laumon, Transformation de Fourier géneralisée, Alg-geom/9603004.
[16] N.C. Leung, S.-T. Yau, E. Zaslow, From special Lagrangian to Hermitian–Yang–Mills via Fourier–Mukai transform, Math. DG/0005118.
[17] G. Marelli, Ph.D. thesis, International School for Advanced Studies, Trieste, 2001.
[18] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. 81 (1981) 153–175.
[19] A. Nakayashiki, Structure of Baker–Akhiezer modules of principally polarized Abelian varieties, commuting partial differential operators and associated integrable systems, Duke Math. J. 62 (1991) 315–358.
[20] M.J. Rothstein, Sheaves with connection on abelian varieties, Duke Math. J. 84 (1996) 565–598.
[21] A. Strominger, S.-T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B 479 (1996) 243–259.